Counterfeiting, Screening, and Government Policy

(Job Market Paper)

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Abstract

We construct a search theoretic model of money in which counterfeit money can be produced at a cost, but agents can screen for fake money also at a cost. Counterfeiting can occur in equilibrium when both costs and the inflation rate are sufficiently low. Optimal monetary policy is the Friedman rule. However, the rationale for the Friedman rule in an economy with the circulation of counterfeit money differs from the conventional mechanism that holds in the model when counterfeiting does not occur. We also study optimal anti-counterfeiting policy that determines the counterfeiting cost and the screening cost.

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1 Introduction

Almost every monetary authority in the world that is responsible for the integrity of its currency has employed, to some extent, a counterfeit deterrence policy to sustain an efficient payment system. However, the implementation of the counterfeit deterrence measures is costly. In order to devise an optimal policy response to counterfeiting, one needs to be aware of the effects of various anti-counterfeiting measures on counterfeiting activities and the welfare loss from counterfeiting. The following questions arise: Under what conditions does counterfeit money exist as an equilibrium outcome? How do deterrence measures affect the counterfeiting environment? How does counterfeiting, or its potential threat, distort economic agents’ behavior and the allocation of resources? Is it optimal to eradicate counterfeiting? It is also essential to be well-informed about the influence of monetary policy because of its effect on the value of money, which, in turn, affects the incentive to counterfeit.

We address these questions by developing a monetary search model in which counterfeiting can occur as an equilibrium outcome. In particular, we introduce costly counterfeiting and costly screening into a search theoretic model of money based on the Lagos and Wright (2005) framework. An asset is necessary for an exchange to take place, and the only asset is fiat money supplied by the government. One of the key elements in the model is that agents, at a cost, can produce fake money that is indistinguishable from genuine money. The other key assumption is that agents can screen out counterfeits at a positive cost. These two conflicting decisions provide a rich framework for understanding counterfeiting and its influence on the economy.

A key insight of our model is that counterfeiting occurs if and only if there is a screening activity. The intuition for this result is as follows. If an agent produces counterfeits for exchanges and money is not screened, the agent can always find a profitable deviation that prevents counterfeiting from taking place by reducing the quantity of the money transfer.\(^1\) However, preventing counterfeiting by restricting the money transfer is costly because it limits the volume of exchanges.

\(^1\)This is why it was difficult to generate counterfeiting in equilibrium in previous models that studied counterfeiting without a screening decision.
Screening can relax this constraint on the money transfer and the exchange volume increases, but for an agent to find it optimal to screen, there must be some counterfeits in the economy.

Equilibrium can be one of three types: no threat of counterfeiting equilibrium, threat of counterfeiting equilibrium, and counterfeiting equilibrium. First, in the no threat of counterfeiting equilibrium, the counterfeiting cost is so high that there is no incentive to produce fake money. Thus, economic activities are the same as in an economy where counterfeiting is not a possibility. Second, in the threat of counterfeiting equilibrium, the counterfeiting cost is not too high but the screening cost is relatively high, so no agents screen money in a trade to check its authenticity. Therefore, counterfeits do not exist in this equilibrium. However, the counterfeiting cost matters for real allocations because it restricts the volume of exchange, as in Li and Rocheteau (2011) and Shao (2014). Finally, when both the counterfeiting cost and the screening cost are sufficiently low, the economy is in the counterfeiting equilibrium where both counterfeiting and screening occur, so genuine and counterfeit monies coexist.

In the last two equilibria, counterfeiting, or the threat of counterfeiting, generates a distortion in the allocation. On the one hand, the quantity traded is inefficiently small in the threat of counterfeiting equilibrium because of the restriction on money transfers. On the other hand, the quantity traded is larger in the counterfeiting equilibrium than the one in the economy without counterfeiting possibility because of the pecuniary effect of increasing trade volume.

One of key messages of our analysis is that the inflation rate plays a critical role in which equilibria exist, unless the counterfeiting cost is too high, and it is more likely that counterfeits circulate in the economy with low inflation. When inflation is high enough, the value of money is low so that agents have no incentive to produce bogus money (i.e., no threat of counterfeiting equilibrium). As inflation falls, the incentive compatibility constraint which guarantees that agents do not produce fake money without screening becomes tighter. However, if inflation is not too low, this incentive constraint is still satisfied via binding constraint, so counterfeiting does not occur in equilibrium (i.e., threat of counterfeiting equilibrium). Finally, when inflation is sufficiently low, the incentive compatibility constraint no longer holds, so counterfeits circulate in the economy
Surprisingly, lowering the inflation rate in the *counterfeiting equilibrium* reduces counterfeiting activities as a result of the strategic behavior of the agents.

Finally, we extend the model to study optimal government policies: monetary and anti-counterfeiting policies. Monetary policy determines the inflation rate through changing the money supply. Anti-counterfeiting policy determines the counterfeiting environment, i.e., the counterfeiting cost and the screening cost, by investing resources in counterfeit deterrence measures. For the detailed study of anti-counterfeiting policy, we introduce three types of counterfeit deterrence measures into the model to capture practices in the real world. The first measure increases the counterfeiting cost and decreases the screening cost. One good example is embedding security features into banknotes. A visible security feature in banknotes that is difficult to reproduce serves as the most obvious means of authentication, and it makes counterfeiting harder. The second measure only increases the counterfeiting cost. For example, a group of countries has developed a system that prevents the reproduction of their banknote images by counterfeiters using personal computers or digital imaging tools. This system makes counterfeiting harder but does not facilitate the screening process. Finally, the third measure acts to decrease the screening cost. Education programs to acquaint the general public with security features and installing authentication devices at sales locations can be examples of the third measure.

One important result of the welfare analysis is that welfare decreases with inflation, regardless of counterfeit deterrence policy, while monetary policy has different effects on the economy, depending on the counterfeiting environment. This directly implies that optimal monetary policy is a Friedman rule, i.e., contracting the money supply at a rate equal to the agent’s rate of time preference. In a standard money search model, the Friedman rule is optimal because it corrects the monetary distortion and thus supports an efficient amount of trade in the economy. This mechanism still works in our model as long as counterfeiting does not occur in equilibrium. A novelty here is that the Friedman rule does not maximize the trade surplus as in a standard Lagos and

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2A counterfeit deterrence system (CDS) has been developed by the Central Bank Counterfeit Deterrence Group (CBCDG) to deter the use of personal computers, digital imaging equipment, and software in the counterfeiting of banknotes in 2004. For more information, visit their official website (http://www.rulesforuse.org).
Wright (2005) model if counterfeits circulate in the economy. Indeed, when the inflation rate is close to the rate of time preference in the *counterfeiting equilibrium*, there is an over production and trade above the efficient level, so that lowering the inflation rate reduces the trade surplus. The rationale behind the Friedman rule here, is that it minimizes counterfeiting activity and therefore also minimizes its welfare costs.

In addition to monotonicity, welfare increases discontinuously when the economy switches from the *threat of counterfeiting equilibrium* to the *counterfeiting equilibrium*. This discontinuity implies that if equilibria with and without counterfeiting are possible under the same economic conditions, then welfare is higher in the economy with circulation of counterfeits than without. This is because when both equilibria are possible, the quantity traded without counterfeiting is too small.

For anti-counterfeiting policy, the model suggests that the government should focus on improving only one dimension: either increasing the counterfeiting cost or reducing the screening cost. Thus, if the main goal of the government is to make counterfeiting hard, for example, it should not invest in the third measure that only facilitates the screening process. The dimensions that the government must pursue and the exact form of its optimal anti-counterfeiting policy depend on technology and preferences. This implies that counterfeit money can exist even under optimal anti-counterfeiting policy if the government can reduce the screening cost more effectively than increasing the counterfeiting cost.

We are certainly not the first to study counterfeiting in the context of monetary search theory. Kultti (1996) and Green and Weber (1996) are earlier works that studied counterfeiting in the context of a monetary search model with indivisible money. Williamson (2002) and Nosal and Wallace (2007) extended previous papers and showed that counterfeiting is only a threat that does not occur in equilibrium, but such a threat could potentially lead to the collapse of a monetary equilibrium. Li and Rocheteau (2011) modify Nosal and Wallace (2007) and show that a monetary equilibrium always exists, but the threat of counterfeiting affects the real allocation. Shao (2014) extends the previous literature by using divisible money and shows that the threat of counterfeiting...
generates an endogenous resalability constraint under competitive price posting. In contrast to other papers, Monnet (2005) and Cavalcanti and Nosal (2011) adopt a mechanism design approach to study the effects of a counterfeiting environment on allocations.

This paper contributes to the literature in three respects. First, our model goes beyond earlier models by endogenizing verification efforts to detect counterfeits with a costly screening process instead of assuming fixed signals of the authenticity of money. Fung and Shao (2016) take a costly verification technology that is similar to ours in spirit into account in their model to endogenize detection efforts. However, they assume that sellers can either invest in the verification technology or not, excluding stochastic investment decisions that contributed to the non-existence problem of monetary equilibrium when the inflation rate and the verification cost are sufficiently high. By allowing stochastic screening, we show that monetary equilibrium always exists.

Second, in our model, both types of equilibria with and without counterfeiting exist making it possible to study how economic factors, such as monetary and anti-counterfeiting policies, affect a counterfeiting state in the economy. In contrast to our result, counterfeiting does not occur in equilibrium in Williamson (2002), Nosal and Wallace (2007), Li and Rocheteau (2011), and Shao (2014), or the models focused only on equilibrium with counterfeiting as in Kultti (1996), Green and Weber (1996), and Fung and Shao (2016). These works cannot rationalize some of stylized facts about counterfeiting; for example, why counterfeiting is a problem in some countries but fraud is not present (or negligible at least) in other countries?

Third, we explore the effects of monetary and anti-counterfeiting policies comprehensively to characterize optimal government policy. Cavalcanti and Nosal (2011) study optimal allocation by using a mechanism design approach, and Lengwiler (1997) analyzes the optimal security level of banknotes using a game theoretic model involving the central bank and a counterfeiter. However, those researchers do not study how monetary and anti-counterfeiting policies interact with each other to find optimal policy.

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3Quercioli and Smith (2015) also use a costly verification effort to study counterfeiting in a game-theoretic model.

4In the extension of Shao (2014) counterfeiting either exists or not depending on parameter values. However, he makes contracts incomplete, non-contingent on buyer’s type, in the extension to make possible that counterfeiting occurs in equilibrium.
The rest of the paper is organized as follows. Section 2 presents the environment of the baseline model and section 3 contains the construction and analysis of equilibrium. In section 4 we extend the model to study optimal government policy. Section 5 is the conclusion.

2 Environment

The general framework is built on Lagos and Wright (2005) with heterogeneous agents similar to Lagos and Rocheteau (2005) and Rocheteau and Wright (2005) incorporating the counterfeiting technology from Li, Rocheteau, and Weill (2012) and the screening technology. Time is indexed by \( t = 0, 1, 2, \ldots \), and there are two subperiods within each period; the centralized market (CM) followed by the decentralized market (DM).

There are two types of economic agents, each with unit mass: buyers and sellers. Each buyer has preferences given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t [X_t - H_t + u(x_t)],
\]

and each seller has preferences given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t [X_t - H_t - h_t].
\]

Here, \( \beta \in (0, 1) \) is the discount rate, \( X_t \) and \( H_t \) are consumption and labor supply, respectively, in the CM, \( x_t \) is consumption in the DM, and \( h_t \) is labor supply in the DM. We assume that \( u(\cdot) \) is a strictly increasing, strictly concave, and twice continuously differentiable function with \( u(0) = 0, u'(0) = \infty, u'(\infty) = 0, -x u''(x) u'(x) < 1 \) for all \( x \geq 0 \), and with the property that there exists some \( \hat{x} \) such that \( u(\hat{x}) = \hat{x} \). Define \( x^* \) by \( u'(x^*) = 1 \).

Each agent can produce one unit of the perishable consumption good for each unit of labor supply. Notice that buyers want to consume but cannot produce in the DM while sellers can produce but do not wish to consume in the DM, which generates a double coincidence problem. The CM is a centralized Walrasian market in which agents trade numeraire CM goods and an asset.
There are bilateral meetings between buyers and sellers in the *DM*. In this economy there is no memory or recordkeeping, so that in any meeting, the traders have no knowledge of each other’s histories. Also no one can be forced to work, so lack of memory implies that there can be no unsecured credit. Hence, an asset is essential for trade to occur. The only asset in this economy is fiat money which is traded at the price $\phi_t$ in terms of numeraire goods in the *CM* in period $t$. Money is supplied by the government at the beginning of the *CM* with a lump-sum transfer $T_t = (\mu - 1)\phi_t M_{t-1}$ to each buyer. Thus, the money stock grows at the constant gross rate $\mu$. In a class of explicit models of money, “New Monetarist Economics” in the language of Williamson and Wright (2010, 2011), with divisible money, $\frac{\phi_t}{\phi_{t+1}} \geq \beta$ must hold.\(^5\) Otherwise, equilibrium does not exist. Thus, below, we consider the case where $\frac{\phi_t}{\phi_{t+1}} \geq \beta$, and when $\frac{\phi_t}{\phi_{t+1}} = \beta$ we only consider the limiting equilibrium as $\frac{\phi_t}{\phi_{t+1}} \to \beta$.\(^6\)

A key assumption is that a buyer can produce any quantity of fake money if he or she incurs a fixed cost of $k$ in the *CM*.\(^7\) Counterfeit money is indistinguishable from genuine money in the *DM*. However, we assume that money is perfectly recognizable in the *CM*, so all counterfeits are confiscated in the *CM*.\(^8\) As a result, there is no incentive for sellers to receive counterfeits in a *DM* meeting. On the other hand, we assume that sellers have a screening technology that can detect counterfeit money with $\gamma$ units of labor in the *DM*.\(^9\) At this moment, we treat $k$ and $\gamma$ as parameters.

\(^5\)New Monetarist approach is also surveyed in Lagos, Rocheteau, and Wright (forthcoming), and a textbook treatment is in Nosal and Rocheteau (2011).

\(^6\)Later, we will focus on stationary equilibrium in which $\frac{\phi_t}{\phi_{t+1}} = \mu$, so the inflation rate is determined by the money growth rate.

\(^7\)Think of the cost of counterfeiting $k$ as the cost of acquiring a sophisticated reprographic machine and photo editing software.

\(^8\)We make this assumption for the following reasons. First, this assumption is used in the existing literature; for example Nosal and Wallace (2007) and Shao (2014). By having an economic environment as close as possible, we can better compare our model with existing studies and highlight the screening technology as a key for generating different results. Second, if counterfeits are not recognizable in the *CM* and buyers make counterfeits in equilibrium, then buyers would produce an infinite amount of counterfeits in a given period to use them for trades in both the *CM* and *DM* (if possible) in all periods, which could threaten the existence of monetary equilibrium. Finally, notice that it is weakly optimal for a buyer to produce an infinite quantity of counterfeits once he incurs the fixed cost $k$. Thus, without the 100% confiscation of counterfeits, there would be two types of buyers when counterfeits are produced: one with counterfeits from the previous period, and the other without any counterfeits from the previous period. This would generate another signaling problem in addition to the one we explore in this paper, which complicates the analysis. The limited ability of the government to take counterfeits out of circulation can provide interesting new insights, but we leave this to future research.

\(^9\)One way to think of the screening cost $\gamma$ is that it reflects the time and effort to scrutinize security features without error. Another interpretation is the cost of installing an authentication device and the effort to use it.
and we endogenize them in section 4 where we discuss optimal government policy.

In this economy, the sequence of moves in each period is as follows: 1) At the beginning of the CM in period \( t \), the government transfers or taxes money to buyers in a lump-sum way. 2) The buyer chooses a portfolio of \( m_t \) genuine money and \( m_t^c \) counterfeit money in terms of period \( t+1 \) CM goods. 3) In the DM, the buyer is matched with a seller and makes a take-it-or-leave-it offer \((x_t, d_t)\) that specifies the quantity of DM goods produced by the seller \( x_t \) and the money transfer \( d_t \) in terms of CM goods in period \( t+1 \) from the buyer to the seller. 4) The seller decides whether to accept the offer or not. 5) If the seller accepts the offer, the buyer hands over \( \hat{d}_t \) genuine money and \( \hat{d}_t^c \) counterfeit money with \( \hat{d}_t + \hat{d}_t^c = d_t \). 6) Then, the seller decides whether to screen the money. If the seller detects counterfeits through screening, then the seller makes new offer \((x_s, d_s)\) over DM goods production and legal money transfer, and the buyer decides whether to accept the seller’s offer or not. Otherwise, the trade goes through according to the proposed offer by the buyer.

### 3 Equilibrium

#### 3.1 Payoffs

To set the stage for equilibrium characterization, we first describe agents’ payoff in the game in a given period \( t \). Because of the quasi-linearity of preferences in Lagos and Wright models, the agent’s value in the CM is linear in his wealth and the choice of asset portfolio is independent of his initial wealth when he entered the period. Then, the buyer’s payoff in the game given the sequence of moves described above is:

\[
S^b_t = - \left( \frac{\phi_t}{\phi_{t+1}} - \beta \right) m_t - k I_{\{m_t^c > 0\}} + I_{\{r = A\}} \left\{ (1 - I_{\{\hat{d}_t^c > 0\}}) [u(x_t) - \beta d_t] + I_{\{s = Y\}} I_{\{r = A\}} [u(x_s) - \beta d_s] \right\},
\]

where \( I_{\{m_t^c > 0\}}, I_{\{r = A\}}, I_{\{\hat{d}_t^c > 0\}}, I_{\{s = Y\}}, \) and \( I_{\{r = A\}} \) are all indicator functions that equal one if \( m_t^c > 0 \), the seller accepts the buyer’s offer \((r = A)\), the buyer hands over a positive amount of
counterfeits to the seller \((\hat{d}_t^c > 0)\), the seller screens the money \((s = Y)\), and the buyer accepts the seller’s offer once counterfeits are detected by the seller \((r^b = A)\), respectively.

The term \((\frac{\phi}{\phi_{t+1}} - \beta) m_t\) is the cost of holding \(m_t\) legal money while in order to produce \(m_t^c > 0\) counterfeits, the buyer must incur the fixed cost \(k\). The terms in the big curly bracket is the payoff from a trade with a seller. If the buyer does not hand over any counterfeits, i.e., \(d_t = \hat{d}_t\), then the seller produces \(x_t\) units of \(DM\) goods with certainty once he accepts the offer. However, if the buyer hands over some counterfeits \(\hat{d}_t^c\), then the terms of trade depend on whether the seller screens the money or not as well as the quantity of genuine money that the buyer transferred, \(\hat{d}_t\). First, if the seller screens the money and detects counterfeits, then he makes a new offer \((x_t^s, d_t^s)\) to the buyer where \(d_t^s \leq \hat{d}_t\). However, if the seller does not screen, then the buyer consumes \(x_t\) units of \(DM\) goods for the exchange of \(\hat{d}_t\) units of legal money.

In a bilateral meeting, the seller’s money holding does not affect any outcome of the game, so, given the money holding costs, the seller would not bring any money into the \(DM\). Then, the seller’s payoff when he meets a buyer who offers \((x_t, d_t)\) can be written as

\[
S^s_t = I_{\{r = A\}} \left\{ I_{\{s = Y\}} \left\{ (1 - I_{\{\hat{d}_t^c > 0\}})(-x_t + \beta d_t) \right. \right.
+ \left. I_{\{\hat{d}_t^c > 0\}}I_{\{r^b = A\}}(-x_t^s + \beta d_t^s) - \gamma \right. \right.
+ \left. (1 - I_{\{s = Y\}})(-x_t + \beta \hat{d}_t) \right\}.
\]

Here, after accepting the buyer’s offer, if the seller screens the money, he will trade \(DM\) goods only for the exchange of genuine money, but he has to pay the screening cost \(\gamma\). On the other hand, if he does not screen the money, then the trade occurs according to the buyer’s offer \((x_t, d_t)\), but he receives \(\hat{d}_t\) units of legal money that can be equal to or less than \(d_t\).

### 3.2 Equilibrium concept

Our definition of equilibrium is standard: given prices, all agents behave optimally and all markets clear in equilibrium. However, the environment, as discussed in the previous section, generates a game between the buyer and the seller, and hence we need more delicate solution concepts for the game, which are discussed in this subsection. Here, we adopt Perfect Bayesian Equilibrium (PBE)
as our equilibrium concept to the game: actions are sequentially rational given a system of beliefs, and beliefs are derived from equilibrium strategies through Bayes’ rule whenever possible.

Before describing agents’ strategies and beliefs, we can simplify a buyer’s actions for portfolio choice using the following logic. After detecting counterfeits, the seller makes a take-it-or-leave-it offer to the buyer, and there is no asymmetric information on the authenticity of the money on the table, which implies $u(x^*_t) - \beta d^*_t = 0$. Then, given $u(x^*_t) - \beta d^*_t = 0$, mixing counterfeits and genuine money does not improve the buyer’s payoff because of the fixed cost of counterfeiting and money holding costs. This argument is formally stated in the next lemma.

**Lemma 1** In any equilibrium, either 1) $d_t = \hat{d}_t \leq m_t$ and $m^*_c = 0$ or 2) $d_t = \hat{d}^*_c \leq m^*_t$ and $m_t = 0$.

**Proof.** See Appendix

Lemma 1 basically means that the buyer will either produce counterfeits or hold legal money only; i.e., he will not hold both legal money and counterfeits in the DM. This implies that if the seller detects any counterfeits in the meeting, there will be no trade between the buyer and the seller, i.e., $x^*_t = d^*_t = 0$. Thus, in the following analysis, we eliminate dominated actions and modify the set of agents’ actions in the game such that: 1) the buyer chooses his portfolio $(m_t, m^*_c)$ from the set $\mathcal{M} = \{(m, 0) | m \in \mathbb{R}_+\} \cup \{(0, m^c) | m^c \in \mathbb{R}_+\}$ in the CM; 2) if the seller detects counterfeits through screening, the trade is not implemented.

Now, in the game at a given period $t$, behavior strategies for a buyer include the probability distribution $f_t(m, m^c)$ over buyer’s portfolio choice $(m, m^c) \in \mathcal{M}$, and the probability distribution $\omega_t(\{x, d\} | (m, m^c))$ from which to draw the offer $(x, d) \in \mathbb{R}_+^2$ conditional on $(m, m^c)$. Seller’s behavior strategies are a decision rule $\eta_t : \mathbb{R}_+^2 \to [0, 1]$ that assigns a probability of acceptance to any feasible offers and a decision rule $\pi_t : \mathbb{R}_+^2 \to [0, 1]$ that assigns a probability of screening the money for all feasible offers. Finally, the seller’s belief regarding the buyer’s action about counterfeiting conditional on the offer $(x, d)$ being made is a mapping $\lambda_t : \mathbb{R}_+^2 \to [0, 1]$ that assigns a probability that the seller meets a genuine money holder.
Then, the PBE of the game at period $t$ is a profile of strategies $\{fi, \omega_t, \eta_t, \pi_t\}$ and belief function $\lambda_t$ such that 1) the agents’ strategies are sequentially rational at each information set, and 2) the belief function $\lambda_t$ is derived from the strategy profile through Bayes’ rule whenever possible.

More specifically, the probability distribution $\omega_t(\{x, d\} | (m, m^c))$ over buyer’s offers, following portfolio choice $(m, m^c)$, must be optimal given the seller’s strategies, that is,

\[
\omega_t(\{\cdot, \cdot\} | (m, 0)) \in \arg\max_{\tilde{\omega}(\tilde{x}, \tilde{d}) \in [0, 1]} \left\{ \int_0^m \int_0^\infty \tilde{\omega}(\tilde{x}, \tilde{d}) \left\{ \eta_t \left( \tilde{x}, \tilde{d} \right) \left[ u(\tilde{x}) - \beta \tilde{d} \right] + \beta m \right\} d\tilde{x}d\tilde{d} \right\},
\]

if $m > 0$ and $m^c = 0$, and

\[
\omega_t(\{\cdot, \cdot\} | (0, m^c)) \in \arg\max_{\tilde{\omega}(\tilde{x}, \tilde{d}) \in [0, 1]} \left\{ \int_0^{m^c} \int_0^\infty \tilde{\omega}(\tilde{x}, \tilde{d}) \eta_t \left( \tilde{x}, \tilde{d} \right) \left[ 1 - \pi_t \left( \tilde{x}, \tilde{d} \right) \right] u(\tilde{x})d\tilde{x}d\tilde{d} \right\},
\]

if $m = 0$ and $m^c > 0$. Let $\mathcal{V}^g_t(m)$ and $\mathcal{V}^c_t(m^c)$ be the maximized value of the objective function in (1) and (2), respectively. Then, given the decision rule $\omega_t(\{x, d\} | (m, m^c))$, $\mathcal{V}^g_t(m)$, and $\mathcal{V}^c_t(m^c)$, the probability distribution $f_i(m, m^c)$ over the buyer’s portfolio choice must be optimal, that is,

\[
f_i(\cdot, \cdot) \in \arg\max_{f(m, m^c) \in [0, 1]} \left\{ \int_0^\infty \int_0^\infty \left\{ \tilde{f}(\tilde{m}, 0) \left[ -\frac{\phi_t}{\phi_{t+1}} \tilde{m} + \mathcal{V}^g_t(m) \right] + \tilde{f}(0, \tilde{m}^c) \left[ -k + \mathcal{V}^c_t(m^c) \right] \right\} d\tilde{m}d\tilde{m}^c \right\}.
\]

Similarly, following an offer $(x, d)$, the seller’s decision to accept the offer and to screen the money must be optimal given the buyer’s strategies and the seller’s belief, that is,

\[
(\eta_t, \pi_t) \in \arg\max_{(\tilde{\eta}, \tilde{\pi}) \in [0, 1]^2} \left\{ \tilde{\eta} \left\{ \tilde{\pi} [\lambda_t(x, d) (-x + \beta d) - \gamma] \right\} + (1 - \tilde{\pi}) [\lambda_t(x, d) (-x + \beta d) - (1 - \lambda_t(x, d))x] \right\}
\]

Finally, $\lambda_t(x, d)$ must satisfy

\[
\lambda_t(x, d) = \frac{\int_d^\infty f_i(m, 0) \omega_t(\{x, d\} | (m, 0)) dm}{\int_d^\infty f_i(m, 0) \omega_t(\{x, d\} | (m, 0)) dm + \int_d^{m^c} f_i(0, m^c) \omega_t(\{x, d\} | (0, m^c)) dm^c}
\]

if the denominator is strictly positive.
Because seller’s beliefs are not pinned down off the equilibrium path under PBE in equation (3), the game would generate a plethora of equilibria.\textsuperscript{10} In order to refine the set of equilibria of the game, we adopt the concept of Reordering Invariance (RI) equilibrium proposed by In and Wright (2012) and applied by Li, Rocheteau, and Weill (2012) in an asset exchange model.\textsuperscript{11} This refinement selects the equilibrium outcomes of the original game that are also equilibrium outcomes of a reordered game in which observed actions (terms of trade) are chosen before unobserved actions (portfolio choice) and shares the same reduced normal form as the original game.\textsuperscript{12}

The timing of reordered game is as follows: 1) At the beginning of the CM in period \( t \) after the government transfers, the buyer posts his DM offer \((x_t, d_t)\). 2) The buyer decides whether to accumulate genuine money or produce counterfeits for the trade. 3) In the DM, the buyer is matched with a seller and the seller decides whether to accept or reject the offer. 4) Once the offer is accepted, the remaining procedures are the same as in the original game. The game tree in the Figure 1 depicts the sequence of moves in the reordered game.

Because of the positive legal money holding costs, the buyer will only acquire \( d \) units of real money in the CM whenever he decides to finance the trade \((x, d)\) with legal money, i.e., \( m_t = d \), and he will produce \( m_t^c \geq d \) units of counterfeits if he decides to have the trade \((x, d)\) with counterfeits. Thus, what matters in the buyer’s portfolio choice given an offer \((x, d)\) is whether to acquire legal money or to produce counterfeits. With this observation in mind, buyer’s behavior strategies, in the reordered game, are the probability distribution \( \omega_t(x, d) \) over offers \((x, d) \in \mathbb{R}^2_+ \) and a decision rule \( \alpha_t : \mathbb{R}^2_+ \rightarrow [0, 1] \) that assigns a probability that the buyer accumulates genuine money conditional on \((x, d)\). Seller’s behavior strategies at each information set are the same as the ones in the original game.

\textsuperscript{10}For example, for any offer \((x, d)\) that satisfies 1) \(-\frac{\phi}{\theta + 1} d + u(x) > \text{Max} \{0, -k + u(x)\} \), and 2) \(-x + \beta d > 0\), it can be supported as a part of equilibrium where \( f_t(d, 0) = 1 \) and \( \omega_t (\{x, d\} | (d, 0)) = 1 \) with a belief system \( \lambda_t \) such that \( \lambda_t(x, d) = 1 \) and \( \lambda_t(x', d') = 0 \), that implies \( \eta_t(x', d') = 0 \), for all \((x', d') \neq (x, d)\).

\textsuperscript{11}In our economic environment, it is hard to apply standard refinement rules in signaling games such as the intuitive criterion because the buyer’s type is chosen endogenously instead of being determined exogenously by nature, so there are additional ways to deviate that must be unprofitable in equilibrium.

\textsuperscript{12}Reversing the order of the game does not affect equilibrium outcomes because the buyer does not gain any pay-off relevant information between his unobserved and observed actions. Reordering Invariance has a strong game theoretic rationale and desirable normative properties. See In and Wright (2012) for more information.
Note, in the reordered game, that any posted offer \((x, d)\) generates a proper subgame denoted by \(\Gamma_t(x, d)\) in Figure 1. Then, by subgame perfection, agents’ strategies restricted to this subgame following an offer \((x, d)\) must form a Nash Equilibrium, so that the seller can correctly infer the buyer’s strategy \(\alpha_t(x, d)\). This implies that we can discipline sellers’ beliefs following all out-of-equilibrium offers \((x, d) \in \mathbb{R}_+^2\) in a logically consistent way as \(\lambda_t(x, d) = \alpha_t(x, d)\).

We now formally state the conditions that agents’ strategies must satisfy in the reordered game. If there is no risk of confusion, we drop arguments for each decision rule from now on; we use \(\alpha_t\) instead of \(\alpha_t(x, d)\), for instance. First, given the offer \((x, d)\) and the seller’s strategies regarding acceptance and screening, the buyer must minimize the cost of financing his DM trade, that is,

\[
\alpha_t \in \arg\min_{\tilde{\alpha} \in [0, 1]} \left\{ \tilde{\alpha} \left[ \frac{\phi_t}{\phi_{t+1}} - (1 - \eta_t)\beta \right] d + (1 - \tilde{\alpha}) \left[ k + \eta_t \pi_t u(x) \right] \right\}.
\]

Here, the term \(\left[ \frac{\phi_t}{\phi_{t+1}} - (1 - \eta_t)\beta \right] d\) is the financing cost with genuine money: Holding cost, \(\left( \frac{\phi_t}{\phi_{t+1}} - \beta \right) d\), plus the expected transferring cost, \(\eta_t \beta d\). The term \(k + \eta_t \pi_t u(x)\) relates to the financing cost with counterfeiting that consists of the fixed cost of producing counterfeits, \(k\), and the expected cost of missing trade by screening in the DM, \(\eta_t \pi_t u(x)\).
Second, the seller’s decision about acceptance and screening must be optimal given the offer \((x,d)\) and the buyer’s strategy, that is,

\[
(\eta_t, \pi_t) \in \arg \max_{(\eta, \pi) \in [0,1]^2} \left\{ \tilde{\eta} \left[ -x + \alpha_t \beta d + \tilde{\pi} \left( (1 - \alpha_t) x - \gamma \right) \right] \right\},
\]

where we imposed the condition that \(\lambda_t = \alpha_t\). The seller’s payoff has two components. The first term, \(-x + \alpha_t \beta d\), is the expected payoff from trade without screening, and the second term, \(\tilde{\pi} \left( (1 - \alpha_t) x - \gamma \right)\), is the net payoff from screening.

Finally, given equilibrium decision rules \(\{\alpha_t, \eta_t, \pi_t\}\), the buyer’s optimal offer maximizes his expected payoff, that is,

\[
(x_t, d_t) \in \arg \max_{(x, d) \in \mathbb{R}_+^2} \left\{ \alpha_t \left[ \frac{-\phi_t}{\phi_t + 1} \tilde{d} + \eta_t \left( u(\tilde{x}) - \beta \tilde{d} \right) + \beta \tilde{d} \right] \right. + \left. (1 - \alpha_t) \left[ -k + \eta_t (1 - \pi_t) u(\tilde{x}) \right] \right\}.
\]

Even though reordering of the game narrows down equilibria of the original game by disciplining the seller’s belief \(\lambda_t\) in an effective way, reordering itself does not guarantee a unique equilibrium outcome in general. In particular, in our model, given \((x,d)\), there could be multiple Nash equilibria of the subgame \(\Gamma_t(x,d)\), which affects, in turn, a buyer’s optimal posting decision.

It is useful to select among equilibria by restricting attention to Pareto dominant Nash equilibria of \(\Gamma_t(x,d)\): there is no other Nash equilibrium that makes every player at least as well off and at least one player strictly better off.\(^1\) Then, given the Pareto dominant Nash equilibrium conditions of the subgame \(\Gamma_t(x,d)\) following all feasible offers \((x,d)\), the buyer posts an offer \((x,d)\) so as to maximize his payoff.

\(^1\)In Li, Rocheteau, and Weill (2012) where there is no screening technology, they did not make any assumptions about selecting equilibrium in the subgame. The difference in their model is that there exists only a pure strategy Nash equilibrium in the subgame given the optimal offer. Thus, by perturbing the offer slightly, they can make a sequence of offers such that all incentive constraints are slack, so the Nash equilibrium of the subgame following perturbed offers is unique and the sequence of offers converges to the equilibrium offer. However, in our model, there could be both a mixed and a pure strategy equilibrium following the equilibrium offer, and we cannot apply the argument of Li, Rocheteau, and Weill (2012) when mixed strategy equilibrium exists. Thus, we resort to the Pareto dominance rule instead.
3.3 Characterization of Equilibrium

In this subsection, we characterize stationary equilibrium. By stationarity, we mean that all real quantities are constant over time, and buyers and sellers play the game repeatedly in the stationary economy. This implies that the inflation rate equals the money growth rate, i.e., $\frac{\phi_t}{\phi_{t+1}} = \mu$.

We now show how to characterize an equilibrium by solving an optimization problem. Consider the following auxiliary problem,

\[(P) \quad S^b = \max_{x \geq 0, d \geq 0, \alpha, \eta, \pi} \left\{ \alpha \left[ -\mu d + \eta (u(x) - \beta d) + \beta d \right] + (1 - \alpha) \left[ -k + \eta (1 - \pi) u(x) \right] \right\}

subject to

\[(4) \quad \alpha \in \arg\min_{\tilde{\alpha} \in [0,1]} \{ \tilde{\alpha} \left[ \mu - (1 - \eta) \beta \right] d + (1 - \tilde{\alpha}) \left[ k + \eta \pi u(x) \right] \}

\[(5) \quad (\eta, \pi) \in \arg\max_{(\tilde{\eta}, \tilde{\pi}) \in [0,1]^2} \{ \tilde{\eta} \left[ -x + \alpha \beta d + \tilde{\pi} [(1 - \alpha) x - \gamma] \right] \}.

Note that any \(\{\alpha, [\eta, \pi]\}\) that satisfies (4) and (5) is a Nash equilibrium of \(\Gamma(x,d)\). Therefore, (P) chooses \(\{x, d, \alpha, \eta, \pi\}\) that maximizes the expected surplus of the buyer subject to a set of Nash equilibria \(\{\alpha, [\eta, \pi]\}\) of the subgame \(\Gamma(x,d)\). Therefore, \(S^b\) is an upper bound of the buyer’s payoff in any equilibrium of the game. Hence, if the buyer can attain \(S^b\) by posting \((x, d)\), it must be an equilibrium offer. It will be shown in proposition 1 below that equilibrium can be constructed and characterized by solving (P). As a preliminary step, we provide some of properties of a solution to (P) beforehand with the interpretation of these properties as equilibrium outcomes.

To begin, observe that for any \(x^0 \leq \frac{\beta}{\mu} k\), \((x,d,\alpha,\eta,\pi) = \left( x^0, \frac{\pi}{\pi}, 1, 1, 0 \right)\) satisfies (4) and (5). Then, the buyer’s expected surplus becomes positive with sufficiently small \(x^0\) because \(\lim_{x \to 0} u'(x) = \infty\), which implies that \(S^b\) must be strictly positive. Thus, any solution must feature \(x > 0\) and \(\eta > 0\) because the buyer’s expected surplus is non-positive otherwise. Also, if \(\alpha = 0\) with positive \(x\), then \(\eta = 0\) by (5). Thus, it must be \(\alpha > 0\) for any solution to (P). Given these, next lemma shows an
important property of a solution providing a useful intermediate step to solve \((P)\).

**Lemma 2** Any solution to \((P)\) has following form: either 1) \(\alpha = 1\) and \(\pi = 0\) or 2) \(\alpha \in (0, 1)\) and \(\pi \in (0, 1)\).

**Proof.** See Appendix

The main implication of lemma 2 is that counterfeiting occurs in equilibrium where buyers attain \(S^b\) only if screening activity exists in the economy. To get some intuition about the logic of this result, consider an offer \((x, d)\) that induces \(\alpha \in (0, 1)\) and \(\pi = 0\) in the following subgame \(\Gamma(x, d)\). The indifference condition of the buyer (4) means that \([\mu - (1 - \eta)\beta]d = k\). Now consider a new offer \((x, d')\) with \(d' = \alpha d\). Then, \([\mu - (1 - \eta)\beta]d' < k\), so it is better for the buyer to bring genuine money with certainty, i.e., \(\alpha' = 1\), provided that \(\eta' = \eta\) and \(\pi' = 0\). Furthermore, if \(\alpha' = 1\), then there is no reason for the seller to screen the money so \(\pi' = 0\), and the seller’s payoff does not change, so \(\eta' = \eta\) is still optimal. Therefore, the buyer can increase his payoff by reducing the quantity of money transfer that prevents counterfeiting from taking place while keeping the seller’s expected payoff unchanged. This profitable deviation explains why counterfeiting does not emerge in equilibrium in previous studies such as Li, Rocheteau, and Weill (2012) and Shao (2014) in which there is no screening technology.

It is also clear, from (4) and (5), that, under what conditions, screening can exist with counterfeiting in the economy. As we argued above, for the buyer to trade with a seller in the \(DM\) with a positive probability, he must propose an offer that induces \(\alpha > 0\) that requires

\[
(6) \quad [\mu - (1 - \eta)\beta]d \leq k + \eta\pi u(x).
\]

Without screening, i.e., \(\pi = 0\), the money transfer \(d\) and the output of \(DM\) goods \(x\) that induces \(\eta > 0\) can be very low if the counterfeiting cost \(k\) is too low. By allowing a positive probability of screening by the seller, the buyer can relax this constraint and increase the trade volume in the \(DM\). However, for the seller to find it optimal to screen the money in equilibrium, the buyer must hand over counterfeits with a positive probability to satisfy (5), so (6) must hold with equality.
Lemma 3 For any solution to \((P)\), \(\eta = 1\) and \(x = \alpha \beta d\).

Proof. See Appendix 

Lemma 3 basically means that the buyer makes an offer that is accepted with certainty and the seller earns zero profit in expectation in equilibrium where buyers attain his maximum payoff \(S^b\). The result that \(x = \alpha \beta d\), that means the seller’s expected payoff is zero, is intuitive given that the buyer makes a take-it-or-leave-it offer to the seller. Next for \(\eta = 1\), that means the buyer makes an offer that is accepted with probability one, Li, Rocheteau, and Weill (2012) also show the same result in their model in which there is no screening technology so counterfeiting does not occur in equilibrium. Here, we only provide the intuition for the finding of \(\eta = 1\) when there are counterfeiting and screening in the economy. Suppose \((\alpha, \eta, \pi) \in (0, 1)^3\). In this case, the constraint (6) binds with equality because of the buyer’s indifference condition. A profitable deviation for the buyer consists of increasing the probability of acceptance by the seller, \(\eta' \in (\eta, 1)\), with a change of the screening probability, \(\pi' \in (0, 1)\), such that the constraint (6) still holds with equality. Then, \(\{\eta', \alpha, \pi'\}\) satisfies (4) and (5) but the buyer’s payoff increases with the higher probability of acceptance of the offer by the seller.

Given lemma 2 and 3, we can rewrite the problem \((P)\) as

\[
(P') \quad S^b = \max_{x \geq 0, d \geq 0, \alpha, \pi} \{-\mu d + u(x)\}
\]

subject to

\[
(7) \quad x = \alpha \beta d
\]

\[
(8) \quad \alpha \in \arg \max_{\tilde{\alpha} \in [0, 1]} \{\tilde{\alpha} [-\mu d + k + \pi u(x)]\}
\]

\[
(9) \quad \pi \in \arg \max_{\tilde{\pi} \in [0, 1]} \{\tilde{\pi}[(1 - \alpha)x - \gamma]\}.
\]

\[\text{14} \text{Notice that by the assumption that } (\alpha, \eta, \pi) \in (0, 1)^3, (1 - \alpha)x = \gamma, \text{ so changing } \pi \text{ does not affect the seller’s expected payoff under this deviation, and } \eta' \in (0, 1) \text{ is still optimal.}\]
Because $-\mu d + u(\alpha \beta d)$ is maximized with $d = \frac{1}{\beta} u^{-1} \left( \frac{\mu}{\beta} \right)$ and $\alpha = 1$, whenever this is feasible, it must be a solution. This is possible if and only if $k \geq \frac{\mu}{\beta} u^{-1} \left( \frac{\mu}{\beta} \right)$, and the solution to $(P')$ is $x = u^{-1} \left( \frac{\mu}{\beta} \right), \ d = \frac{1}{\beta} u^{-1} \left( \frac{\mu}{\beta} \right), \ \alpha = 1, \ \text{and} \ \pi = 0$ in this case.

When $k < \frac{\mu}{\beta} u^{-1} \left( \frac{\mu}{\beta} \right)$, finding a solution of $(P')$ becomes more complicated. However, by virtue of lemma 2, we can restrict our analysis to two cases: 1) $\alpha = 1$ and $\pi = 0$, or 2) $\alpha \in (0, 1)$ and $\pi \in (0, 1)$. We consider these two cases separately and take the max of the two to solve the problem $(P')$.

First, consider the case with $(\alpha, \pi) = (1, 0)$ that requires, from (8), that

(10) \hspace{1cm} d \leq \frac{k}{\mu}.

This is the incentive compatibility constraint for the buyer not to commit forgery without screening. Given $k < \frac{\mu}{\beta} u^{-1} \left( \frac{\mu}{\beta} \right)$, (10) must bind to maximize the objective function of $(P')$, so $x = \frac{\beta k}{\mu}$, $d = \frac{k}{\mu}$. Substituting this candidate solution into the objective function of $(P')$, we obtain $S^b_{ic} \equiv -k + u \left( \frac{\beta k}{\mu} \right) > 0$. Here, the trade volume in the DM is limited by the counterfeiting cost $k$ and the buyer’s surplus $S^b_{ic}$ converges to zero as $k \to 0$.

Second, consider a candidate solution with $\alpha \in (0, 1)$ and $\pi \in (0, 1)$. The conditions (8) and (9) yield a mixed strategy equilibrium only if the following equations hold:

(11) \hspace{1cm} \mu d = k + \pi u(x)

(12) \hspace{1cm} (1 - \alpha) x = \gamma.

As one can see from (10) and (11), a positive probability of screening, $\pi > 0$, allows the buyer to make an offer with the money transfer higher than $\frac{k}{\mu}$, so the buyer could consume more DM goods. However, there is a cost for this strategy. To make the seller to screen the money with a positive probability, the buyer must produce and transfer counterfeits with a positive probability so (12) holds. Then, because the seller receives legal money with probability of $\alpha$, the buyer must hand over $d = \frac{x}{\alpha \beta}$ units of money, that is higher than $\frac{x}{\beta}$, to make the seller earns non-negative profit.
in expectation. Thus, there is additional cost to finance DM trade which is caused by the fraud and screening. Then, the buyer optimally chooses \((x, d, \alpha, \pi)\) considering all those effects on his payoff.

**Lemma 4** Suppose 
\[-\frac{\mu}{\beta \gamma} \frac{x_{\gamma}^2}{x_{\gamma - \gamma}} + u(x_{\gamma}) \geq 0, \]
where \(x_{\gamma} \in \left(u^{-1}\left(\frac{\mu}{\beta}\right), \infty\right)\) is given by
\[
\frac{\mu}{\beta} = \frac{(x_{\gamma} - \gamma)^2}{x_{\gamma}(x_{\gamma} - 2\gamma)} u'(x_{\gamma}).
\]

Then, the candidate solution of \((P')\) where \((\alpha, \pi) \in (0, 1)^2\) is \(x = x_{\gamma}, d = \frac{x_{\gamma}^2}{\beta(x_{\gamma} - \gamma)}, \alpha = 1 - \frac{\gamma}{x_{\gamma}}, \) and 
\[
\pi = \frac{1}{u(x_{\gamma})} \left\{ \frac{\mu}{\beta} \frac{x_{\gamma}^2}{x_{\gamma} - \gamma} - k \right\},
\]
and the value of the objective function is \(S_c^b = -\frac{\mu}{\beta} \frac{x_{\gamma}^2}{x_{\gamma} - \gamma} + u(x_{\gamma}).\)

**Proof.** See Appendix □

In lemma 4, we derive the second candidate solution of \((P')\) only for the case that \(S_c^b \geq 0,\) even though it is possible to have \(S_c^b < 0\) with sufficiently high \(\gamma.\) However, this is without loss of generality in the following sense: \(S_c^b\) is an upper bound of the maximized value of the problem \((P')\) with constraints that \((\alpha, \pi) \in (0, 1)^2\) because we do not use constraints \((11)\) and \(\pi \in (0, 1)\) to get \(S_c^b\) (see the proof for details), so whenever \(S_c^b < 0, (x, d, \alpha, \pi) = \left(\frac{\beta k}{\mu}, \frac{k}{\mu}, 1, 0\right)\) solves \((P').\)

The final step is to compare the buyer’s payoff under each candidate solution to get \(S_c = \max\{S_{tc}^b, S_c^b\}.\) Subtracting \(S_{tc}^b\) from \(S_c^b,\) we obtain
\[
(14) \quad G(\mu, k, \gamma) \equiv -\frac{\mu}{\beta} \frac{x_{\gamma}^2}{x_{\gamma} - \gamma} + u(x_{\gamma}) - \left\{ -k + u\left(\frac{\beta k}{\mu}\right) \right\}.
\]

Thus, if \(G(\mu, k, \gamma) < 0,\) then \((x, d, \alpha, \pi) = \left(\frac{\beta k}{\mu}, \frac{k}{\mu}, 1, 0\right)\) is the solution to \((P').\) On the other hand, if \(G(\mu, k, \gamma) > 0,\) then \((x, d, \alpha, \pi) = \left(x_{\gamma}, \frac{x_{\gamma}^2}{\beta(x_{\gamma} - \gamma)}, 1 - \frac{\gamma}{x_{\gamma}}, \frac{1}{u(x_{\gamma})}, \left\{ \mu \frac{x_{\gamma}^2}{\beta(x_{\gamma} - \gamma)} - k \right\} \right)\) solves \((P').\) Finally, in the knife edge case where \(G(\mu, k, \gamma) = 0, S_{tc}^b = S_c^b,\) so both candidate solutions solve \((P').\)

To gather more intuition about the influence of the counterfeiting environment, \((k, \gamma),\) on the
economy, take derivatives $G$ with respect to $k$ and $\gamma$ to obtain

\[
\frac{\partial G}{\partial k} = 1 - \frac{\beta}{\mu} u'(\frac{\beta k}{\mu}) < 0,
\]
\[
\frac{\partial G}{\partial \gamma} = -\frac{\mu}{\beta} \left( \frac{x_\gamma}{x_\gamma - \gamma} \right)^2 < 0,
\]

where the inequality of the first equation comes from the assumption that $k < \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$, and we used the envelope theorem to obtain the second result. Note that $G(\mu, k, 0) > 0$ and $G(\mu, k, \infty) < 0$ provided that $k < \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$. Taken together with the result of the case where $k \geq \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$, we define new function $\tilde{\gamma}: [\beta, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ such that

(15) $\tilde{\gamma}(\mu, k) = \begin{cases} 
\tilde{\gamma} > 0 \text{ that satisfies } G(\mu, k, \tilde{\gamma}) = 0 & \text{if } k < \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right) \\
0 & \text{if } k \geq \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right). 
\end{cases}$

Then, if $\gamma < \tilde{\gamma}(\mu, k)$, a solution $(x, d, \alpha, \eta, \pi)$ that solves $(P)$ features $\alpha < 1$.

So far, we have characterized the upper bound of payoff attainable by buyers. The last step is to show that equilibrium can be characterized using $(P)$. The basic idea, detailed in the proof, is to show that for any solution $(x, d, \alpha, \eta, \pi)$ of $(P)$, $\{\alpha, [\eta, \pi]\}$ is a unique Pareto dominant Nash equilibrium in the following subgame $\Gamma(x, d)$. Thus, whenever the buyer posts $(x, d)$ that solves $(P)$, he can achieve the maximum payoff $S^b$. This leads to the following proposition.

**Proposition 1** There exists monetary equilibrium that features:

1. **[No threat of counterfeiting]** If $k \geq \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$, then $x = u'^{-1}\left(\frac{\mu}{\beta}\right)$, $d = \frac{1}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$, $\alpha = 1$, $\eta = 1$, and $\pi = 0$

2. **[Threat of counterfeiting]** If $k < \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$ and $\gamma \geq \tilde{\gamma}(\mu, k)$, then $x = \frac{\beta k}{\mu}$, $d = \frac{k}{\mu}$, $\alpha = 1$, $\eta = 1$, and $\pi = 0$

3. **[Counterfeiting]** If $k < \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$ and $\gamma < \tilde{\gamma}(\mu, k)$, then $x = x_\gamma$, $d = \frac{x_\gamma^2}{\beta (x_\gamma - \gamma)}$, $\alpha = 1 - \frac{\gamma}{x_\gamma}$, $\eta = 1$, and $\pi = \frac{1}{u(\gamma)} \left\{ \frac{\mu}{\beta} \frac{x_\gamma^2}{x_\gamma - \gamma} - k \right\}$.\footnote{When the equilibrium type switches from the threat of counterfeiting to the counterfeiting, the set of strategies...}
Proof. See Appendix.

Proposition 1 shows how the counterfeiting environment, \((k, \gamma)\), and the inflation rate, \(\mu\), together determine the existence of particular equilibria. This result is illustrated with Figure 2 that depicts how the parameter space is subdivided with \(k\) on the vertical axis, and \(\gamma\) on the horizontal axis given the inflation rate \(\mu\).

In the no threat of counterfeiting equilibrium, the counterfeiting cost is too high for the buyer to produce fake money. In this case, the incentive compatibility constraint (10) does not bind. The terms of trade \((x, d)\) are the same as the ones in an economy where there is no possibility of counterfeiting when the buyer makes a take-it-or-leave-it offer.

In the threat of counterfeiting equilibrium, a low counterfeiting cost is accompanied with a relatively high screening cost; \(\gamma \geq \bar{\gamma}(\mu, k)\). Because of the low counterfeiting cost, there could exist the incentive for the buyer to produce counterfeits. However, the buyer offers terms of trade that thwart entry in counterfeiting by limiting the money transfer with the binding incentive compatibility constraint (10) instead of making an offer that induces counterfeiting and screening because of the high screening cost. In this case, the usefulness of money as a medium of exchange depends on the counterfeiting cost, \(k\), so the quantity of goods traded, \(x\), is inefficiently low.

\(\{x, d, \alpha, \eta, \pi\}\) changes discontinuously. Thus, in the knife edge case where \(\gamma = \bar{\gamma}(\mu, k)\), the model admits multiple equilibria. However, analysis of multiple equilibria does not give any important insight at this moment, so we assume that the economy is in the threat of counterfeiting equilibrium in this case. Later, we discuss multiple equilibria when we characterize welfare in section 4.
Finally, when both the counterfeiting cost and the screening cost are sufficiently low, the economy is in the *counterfeiting equilibrium*. If \( k \) is very low, the binding constraint (10) without the screening can be too restrictive on the money transfer \( d \). Further, if the screening cost is low such that \( \gamma < \gamma(\mu, k) \) as in this equilibrium, the costs from having a positive probability of frauds and screening are relatively low.\(^{16}\) Thus, it is better for the buyer to make an offer that induces positive probabilities of counterfeiting and screening in the following subgame, so (11) and (12) hold, instead of satisfying the constraint (10). In this case, the quantity of \( DM \) goods traded, \( x \), money transfer, \( d \), and the probability that the buyer hands over genuine money, \( \alpha \), depend on \( \gamma \) but not on \( k \). The counterfeiting cost, \( k \), only affects the probability of screening, \( \pi \). Thus, the model implies that the extent that money facilitates exchange depends on the ease of screening out fake money in the *counterfeiting equilibrium* while it depends on the counterfeiting cost in the *threat of counterfeiting equilibrium*.

One interesting feature in the *counterfeiting equilibrium* is that consumption in the \( DM \) is higher than in an economy where counterfeiting was not even a possibility: \( x_\gamma > u^{-1}(\frac{\mu}{\beta}) \) (see lemma 4). This result can be understood by looking at the indifference condition (12). The reason to offer \((x, d)\) that induces a positive probability of screening is to consume higher quantity of \( DM \) goods \( x \). According to (12), higher \( x \) increases \( \alpha \) in the *counterfeiting equilibrium*. The seller can avoid producing \( DM \) goods for nothing if he detects counterfeit money. Thus, if \( x \) increases, the probability that the buyer hands over genuine money must increase to keep the seller indifferent between screening and no screening given the fixed cost of screening \( \gamma \). However, non-negative profit condition of the seller, \( x = \alpha \beta d \), implies that the relative money transfer to the \( DM \) goods produced, \( \frac{d}{x} \), decreases with \( x \). Because of this pecuniary effect of increasing \( x \), the buyer offers \( x_\gamma \) that is strictly higher than \( u^{-1}(\frac{\mu}{\beta}) \).

As one can see from proposition 1, our model admits both equilibria with and without counterfeiting and monetary equilibrium always exists. Thus, the model can explain the cross-country differences in counterfeiting experiences as an equilibrium outcome.\(^{17}\) For example, two coun-

\(^{16}\)Note that as \( \gamma \to 0, \alpha \to 1 \) from (12), so \( d = \frac{\alpha}{\alpha \beta} \to \frac{\gamma}{\beta} \) and the additional cost from frauds disappears.

\(^{17}\)For example, counterfeiting is a problem in some countries while it is not in other countries (see Fung and Shao.
tries with the same inflation rate could have different counterfeiting experiences depending on the counterfeiting environment \((k, \gamma)\). These cross-country differences could not be well explained by previous models in which counterfeiting does not occur or always exists in equilibrium.

On a related point, it seems worthwhile to discuss recent work on fraudulent practices in asset markets in the context of asset exchange models, Li, Rocheteau, and Weill (2012) and Shao (2014).\(^{18}\) One of main implications in their research is that the threat of counterfeiting generates an endogenous resalability constraint similar to (10), specifying the asset’s usefulness as a medium of exchange. It is this incentive compatibility constraint that prevents forgery from taking place, and it is so powerful that fraud does not occur in equilibrium even with the counterfeiting cost close to zero.

One important unnoticed assumption, however, in their models is that there is no screening technology, so the only strategy for a seller in response to counterfeiting is rejecting an offer: If trade involves an asset transfer greater than the upper bound specified by the resalability constraint, then the seller would reject the offer with positive probability. However, as argued above, this cannot be optimal for the buyer.

By incorporating a screening technology, our model shows that this resalability constraint can become ineffective thus generating fraud as an equilibrium outcome provided that the screening cost is sufficiently low. The intuition is simple: With the screening technology, the seller has an additional action to react to counterfeiting, and as long as the seller earns non-negative profit \(ex-ante\) with the optimal screening strategy, he would accept the buyer’s offer with certainty even though the resalability constraint is not satisfied. Notice that our model encompasses those previous studies as a special case where the screening cost is sufficiently high.

**Monetary policy and the equilibrium type** So far, we have taken the inflation rate, \(\mu\), as given. We now study how monetary policy that determines \(\mu\) affects economic agents’ behaviors and

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\(^{18}\)Observe that all agents take the inflation rate that is inverse of rate of return on money as given when they make their optimal strategies in the game. Thus, all of previous analysis can be applied to the economy with real assets like Li, Rocheteau, and Weill (2012).

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hence the equilibrium type. Given the assumption that \(-x^{u''(x)}\frac{u'(x)}{u'(x)} < 1\), \(\frac{\mu}{\beta}u^{-1}(\frac{\mu}{\beta})\) is decreasing in \(\mu\). Thus, if \(k \geq x^*\), the economy is in the no threat of counterfeiting equilibrium for all monetary policy \(\mu \geq \beta\) (see the first part of proposition 1).

If \(k < x^*\), there exists unique value \(\mu_k > \beta\) that satisfies

\[
(16) \quad k = \frac{\mu_k}{\beta}u^{-1}(\frac{\mu_k}{\beta}).
\]

Then, for all \(\mu \geq \mu_k\), the no threat of counterfeiting equilibrium exists. On the other hand, when \(\mu < \mu_k\), the economy is in either the threat of counterfeiting equilibrium or the counterfeiting equilibrium. Because \(x_\gamma > \frac{\beta k}{\mu}\) in this case, we obtain, from (14),

\[
\frac{\partial G}{\partial \mu} = \frac{1}{\mu} \left\{ -\frac{x_\gamma - \gamma}{x_\gamma - 2\gamma}x_\gamma u'(x_\gamma) + \frac{\beta k}{\mu}u'(\frac{\beta k}{\mu}) \right\} < 0,
\]

so it is more likely that the counterfeiting equilibrium exists as \(\mu\) decreases. More precisely, a decrease in \(\mu\) shifts the horizontal line at point \(\frac{\mu}{\beta}u^{-1}(\frac{\mu}{\beta})\) and the curve given by \(\hat{\gamma}(\mu, k)\) in Figure 2 upward, so the parameter space \((\gamma, k)\) for the counterfeiting equilibrium to exist expands.

To facilitate the presentation of the effects of \(\mu\) on the equilibrium type, we derive new critical value of \(\mu\) in a following way. Note, from (15), that for all \(k < x^*\), there exists unique \(\hat{\gamma}(\beta, k) > 0\) such that \(G(\beta, k, \hat{\gamma}(\beta, k)) = 0\). Then because \(G(\mu, k, \gamma)\) is decreasing in each argument there exists unique \(\mu_\gamma \in (\beta, \mu_k)\), for all \(\gamma < \hat{\gamma}(\beta, k)\), that satisfies

\[
(17) \quad G(\mu_\gamma, k, \gamma) = 0.
\]

Thus, \(G(\mu, k, \gamma) > 0\) for all \(\mu \in [\beta, \mu_\gamma]\), and \(G(\mu, k, \gamma) \leq 0\) for all \(\mu \in [\mu_\gamma, \mu_k]\). Notice that the critical value \(\mu_\gamma\) is a decreasing function of \(k\) and \(\gamma\) because of the property of \(G(\mu, k, \gamma)\). On the other hand, if \(\gamma \geq \hat{\gamma}(\beta, k)\), then \(G(\mu, k, \gamma) \leq 0\) for all \(\mu \in [\beta, \mu_k]\). In summary, we have the following proposition, whose proof is omitted, that describes how monetary policy determines the equilibrium type given \(k\) and \(\gamma\).
Proposition 2 1. Suppose $k \geq x^*$. Then, the no threat of counterfeiting equilibrium exists for all $\mu \geq \beta$.

2. Suppose $k < x^*$ and $\gamma \geq \hat{\gamma}(\beta, k)$. Then, i) the no threat of counterfeiting equilibrium exists for all $\mu \geq \mu_k$, and ii) the threat of counterfeiting equilibrium exists for all $\mu \in [\beta, \mu_k]$.

3. Suppose $k < x^*$ and $\gamma < \hat{\gamma}(\beta, k)$. Then, i) the no threat of counterfeiting equilibrium exists for all $\mu \geq \mu_k$, ii) the threat of counterfeiting equilibrium exists for all $\mu \in [\mu_\gamma, \mu_k]$, and iii) the counterfeiting equilibrium exists for all $\mu \in [\beta, \mu_\gamma]$.

Figure 3 illustrates graphically proposition 2.\textsuperscript{19} As one can see from proposition 2 and Figure 3, it is more likely that counterfeits circulate in the economy with low inflation. Put differently, for any counterfeiting environment, $(k, \gamma)$, there exists a cutoff inflation rate $\mu_\gamma$ (or $\beta$), such that if the inflation rate is above this cutoff value, counterfeits do not exist. The intuition for this finding is as follows. When inflation is high, the cost of holding money is high so quantity of goods traded in the DM is small with genuine money. In this case, counterfeiting would be unprofitable because of its fixed cost. However, as inflation falls, the buyer can finance more DM goods with genuine money, which induces a higher incentive to produce fake money, and finally counterfeiting occurs when inflation is sufficiently low provided that $\gamma < \hat{\gamma}(\beta, k)$.

\textsuperscript{19}In the left panel, $\overline{\gamma}$ satisfies $\overline{\gamma}(\beta, \overline{\gamma}) = \gamma$ given the assumption that $\gamma < \hat{\gamma}(\beta, k = 0)$, and we draw the right panel with the assumption that $k < x^*$. 

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**Equilibrium Comparative Statics** Propositions 1 and 2 provide comprehensive analysis of conditions under which counterfeits circulate or not in the economy. To gather more intuition for the effects of the counterfeiting environment, \((k, \gamma)\), and the inflation rate, \(\mu\), on the economy, we investigate comparative statics with respect to these variables. Because the analysis for the other two cases are straightforward, we focus on the comparative statics in the counterfeiting equilibrium.

First, an increase of inflation lowers \(x\) and \(d\) because of its influence on the money holding cost (see lemma 4). Given this result, the probability of holding genuine money, \(\alpha\), must fall with respect to \(\mu\) in order to satisfy the seller’s indifference condition (12). An increase in \(\mu\) has an ambiguous effect on the probability of screening, \(\pi\).\(^{20}\)

Second, the counterfeiting cost, \(k\), has no influence on terms of trade \((x, d)\), and the probability of accumulating genuine money, \(\alpha\). It only affects the probability of screening, \(\pi\), via the buyer’s indifference condition (11), and \(\pi\) decreases with \(k\), which is intuitive.

Finally, the effects of \(\gamma\) on equilibrium outcomes require more detailed analysis. From (13), we obtain

\[
\frac{\partial x_\gamma}{\partial \gamma} = \frac{2\gamma x_\gamma u'(x_\gamma)}{2\gamma^2 u'(x_\gamma) - x_\gamma(x_\gamma - \gamma)(x_\gamma - 2\gamma)u''(x_\gamma)} > 0.
\]

Next, taking derivatives \(\alpha = 1 - \frac{\gamma}{x_\gamma}\) with respect to \(\gamma\), we get

\[
\frac{\partial \alpha}{\partial \gamma} = \frac{(x_\gamma - \gamma)(x_\gamma - 2\gamma)u''(x_\gamma)}{2\gamma^2 u'(x_\gamma) - x_\gamma(x_\gamma - \gamma)(x_\gamma - 2\gamma)u''(x_\gamma)} < 0.
\]

Thus, an increase of \(\gamma\) raises \(x\) but lowers \(\alpha\). The mechanism for these results can be found from the seller’s indifference condition (12). When \(\gamma\) increases, the buyer must decrease \(\alpha\) or increase \(x\) to satisfy this indifference condition. Even though higher \(x\) means more consumption in the DM, the buyer also has to consider the genuine money transfer, \(d = \frac{x}{\beta}\). Because of the effects on the money transfer, buyers respond to an increase in \(\gamma\) with higher \(x\) but lower \(\alpha\) in the counterfeiting equilibrium. The comparative statics for \(d\) come directly from the comparative statics for \(x\) and \(\alpha\);

\(^{20}\)We conclude this result with numerical simulation. In general when \(\gamma\) is low \(\frac{\partial \pi}{\partial \mu} < 0\) whereas \(\frac{\partial \pi}{\partial \mu} > 0\) when \(\gamma\) is relatively high. However, under some parameter values, \(\pi(\mu)\) is a parabola, i.e., \(\frac{\partial \pi}{\partial \mu} < 0\) for \(\mu \in [\beta, \mu']\) and \(\frac{\partial \pi}{\partial \mu} > 0\) for \(\mu \in [\mu', \mu_\gamma)\).
\[ \frac{\partial d}{\partial \gamma} > 0. \] Last, differentiating \( \pi = \frac{1}{u(x_\gamma)} \left\{ \frac{\mu x_\gamma^2}{\beta (x_\gamma - \gamma)} - k \right\} \) with respect to \( \gamma \), we get

\[
\frac{\partial \pi}{\partial \gamma} = \left(1 - \pi\right) \frac{u'(x_\gamma) \partial x_\gamma}{u(x_\gamma)} + \frac{1}{u(x_\gamma)} \frac{\mu}{\beta} \left( \frac{x_\gamma}{x_\gamma - \gamma} \right)^2 > 0.
\]

This result appears to be counter-intuitive because one would expect that the seller will screen money less with a higher screening cost. However, as long as the seller’s indifference condition (12) is satisfied, changing \( \pi \) does not affect the seller’s expected surplus. \( \pi \) is adjusted to satisfy the buyer’s indifference condition (11) in the counterfeiting equilibrium, and given the effects of changing \( \gamma \) on \( x \) and \( d \), \( \pi \) must rise as \( \gamma \) increases.21

**Inflation rate and counterfeiting** We close this section with a study of a relationship between the inflation rate \( \mu \) and the measure of agents committing fraud, \( 1 - \alpha \). Proposition 2 shows that when \( k < x^* \) and \( \gamma < \gamma(\beta, k) \), the threat of counterfeiting intensifies as inflation falls, and finally counterfeiting materializes in equilibrium since \( \mu \) becomes lower than \( \mu_\gamma \). However, \( \frac{\partial (1 - \alpha)}{\partial \mu} > 0 \) in the counterfeiting equilibrium. Thus, in this environment, the measure of frauds represented by \( 1 - \alpha \) rises as \( \mu \) increases from \( \beta \), but it drops suddenly to 0 once \( \mu \) hits the tipping point \( \mu_\gamma \) as described in Figure 4.22

This feature is useful for understanding the episodic nature of counterfeiting and the unintended results of the anti-counterfeiting policy in Canada.23 These can be explained as an equilibrium out-

21 More precisely, substituting \( d = \frac{x_\gamma^2}{\beta(x_\gamma - \gamma)} \) into (11) and totally differentiating with respect to \( x \) and \( \pi \), we obtain

\[
\left\{ -\frac{\mu x_\gamma (x_\gamma - 2\gamma)}{\beta (x_\gamma - \gamma)^2} + \pi u'(x_\gamma) \right\} \Delta x + u(x) \Delta \pi = 0,
\]

where \( -\frac{\mu x_\gamma (x_\gamma - 2\gamma)}{\beta (x_\gamma - \gamma)^2} + \pi u'(x_\gamma) < 0 \) by (13). Thus, \( \pi \) must increase with \( \gamma \) because \( \frac{\partial x_\gamma}{\partial \gamma} > 0 \).

22 In the discussion of Cavalcanti and Nosal (2011), Monnet (2011) shows positive relationship between inflation and counterfeiting using heterogeneous counterfeiting costs. The difference in his model is that counterfeiting increases monotonically with inflation which implies countries with hyper-inflation suffer the highest counterfeiting. However, it is quite intuitive that no one would struggle to produce fake Reichsmark, old currency of Germany, during the period from 1922 to 1923 when the country went through its worst inflation.

come in our model using the properties that $\frac{\partial \mu}{\partial k} < 0$ and $\frac{\partial \mu}{\partial \gamma} < 0$. Suppose an economy is in the threat of counterfeiting equilibrium with the inflation rate $\mu$ close to $\mu_\gamma$. If there exists a technological innovation that reduces the counterfeiting cost $k$ such that the new critical value $\mu'_\gamma$ is higher than the inflation rate $\mu$, then counterfeits rise sharply in this economy. After the government puts its effort on developing banknotes that are difficult to counterfeit or promoting screening of banknotes by retailers, then counterfeits decrease or disappear from the economy. By the same reasoning, unless the screening cost was cut down significantly such that $\gamma' \approx 0$, an inadequate reduction of $\gamma$ may manifest itself as a sharp increase of counterfeits in this environment.

4 Optimal government policy

In this section, we study optimal government policy that consists of monetary policy and anti-counterfeiting policy. Anti-counterfeiting policy determines the counterfeiting environment $(k, \gamma)$. This is one of main problems that monetary authorities face whenever they develop a new series of banknotes. The government must deliberate on a reasonable trade-off between improved security and the added cost of a counterfeit deterrence measure in order to maximize welfare. In addition to anti-counterfeiting policy, counterfeiting cannot be separated from monetary policy, which affects the value of money. Therefore, these two policies must be taken into account together to find the
optimal government policy.

For this purpose, we endogenize $k$ and $\gamma$ in a following way: The government taxes $\tau_1$, $\tau_2$, and $\tau_3$ to buyers in a lump sum way in the CM of each period and invest in a counterfeit deterrence system to maintain $k = k(\tau_1, \tau_2)$ and $\gamma = \gamma(\tau_1, \tau_3)$. We assume that $k$ and $\gamma$ are twice continuously differentiable functions with each argument, and satisfy $k_i > 0, k_{ii} < 0, \gamma_i < 0,$ and $\gamma_{ii} > 0$ where $k_i = \frac{\partial k(\tau_1, \tau_2)}{\partial \tau_i}$, for instance, with $i \in \{1, 2, 3\}$. Thus, anti-counterfeiting policy aims to improve two dimensions in the counterfeiting environment: An increase of the counterfeiting cost and a decrease of the screening cost. We further assume that $k(0, 0) = 0, \gamma(0, 0) = \infty$, $\lim_{\tau_1 \to \infty} \lim_{\tau_2 \to \infty} k(\tau_1, \tau_2) > x^*$, and $\lim_{\tau_1 \to \infty} \lim_{\tau_3 \to \infty} \gamma(\tau_1, \tau_2) = 0$.

Observe that $\tau_1$ affects both $k$ and $\gamma$. We make this assumption to reflect the nature of counterfeiting deterrence measures because both dimensions are inter-related. For the security features of a banknote to work as a screening device, they must be hard to counterfeit. Therefore, adding new security features to banknotes makes the screening easier while making the production of forged notes harder.

However, there are also anti-counterfeiting measures that focus only on one dimension that is captured by $\tau_2$ and $\tau_3$. Currently, for example, it is no longer possible to reproduce U.S. banknotes with personal computers and digital imaging tools because of a digital watermark that is embedded in banknotes. This system makes counterfeiting much harder because forgers need other machines to make bogus money, but this measure does not improve the screening process. On the other hand, the main purpose of public education about the security features incorporated in banknotes and development of a portable counterfeit detector is to make the screening process easier, but these measures are not related to the production of counterfeits.

To study optimal policy, we need an aggregate welfare measure. If we measure welfare as the sum of expected utilities across agents with equal weight, we obtain

\begin{equation}
W(\mu, \tau) = [1 - \pi(1 - \alpha)][\mu(x) - x] - (1 - \alpha)k(\tau_1, \tau_2) - \pi \gamma(\tau_1, \tau_3) - \tau_1 - \tau_2 - \tau_3,
\end{equation}
where $\tau = (\tau_1, \tau_2, \tau_3)$. As in most Lagos and Wright (2005) setups, $CM$ activities that involve money trades cancel out. However, the labor input to produce fake money in the $CM$ and the labor for screening in the $DM$ do not improve any others consumption, so they only reduce welfare. Also, when counterfeiting and screening exist, missing trade surplus, captured by $\pi(1 - \alpha)[u(x) - x]$, is the deadweight loss of welfare.

As shown in propositions 1 and 2, the counterfeiting cost, $k$, the screening cost, $\gamma$, and monetary policy, $\mu$, interact with each other in determining the equilibrium type, and each element has different effects on the economy depending on the other two elements. Therefore, in order to study optimality, we have to consider all policy measures together. However, understanding how optimal monetary policy depends on the counterfeiting environment is also of interest to the government. Thus, we first take $\tau$ as given, and find $\mu^*(\tau)$ that is obtained by

$$\mu^*(\tau) \in \arg\max_{\mu \geq \beta} \widetilde{W}(\mu; \tau) \equiv [1 - \pi(1 - \alpha)][u(x) - x] - (1 - \alpha)k - \pi\gamma$$

subject to the equilibrium conditions described in proposition 1.\(^\text{24}\) Then, the optimal government policy $(\mu^*, \tau^*)$ is given by $(\mu^*(\tau^*), \tau^*)$ where $\tau^*$ maximizes $\widetilde{W}(\mu^*(\tau), \tau)$ subject to agents’ optimal behaviors.

Since $k \leq x^*$ must hold with optimal government policy, we assume this condition in the following analysis without loss of generality. Suppose first that $\gamma \geq \bar{\gamma}(\beta, k)$. Then, the economy is in either the no threat of counterfeiting equilibrium or the threat of counterfeiting equilibrium depending on $\mu$. Substituting the allocation $(x, d, \alpha, \pi)$ under each equilibrium to $\widetilde{W}(\mu; \tau)$, we obtain

$$\widetilde{W}_{MC}(\mu; \tau) = u\left(u'^{-1}\left(\frac{\mu}{\beta}\right)\right) - u'^{-1}\left(\frac{\mu}{\beta}\right) \text{ for all } \mu \geq \mu_k$$

$$\widetilde{W}_{TC}(\mu; \tau) = u\left(\frac{\beta k}{\mu}\right) - \frac{\beta k}{\mu} \text{ for all } \mu \in [\beta, \mu_k].$$

\(^{24}\)Since we assume that $\tau$ is given at this moment, we use $k$ and $\gamma$ instead of $k(\tau_1, \tau_2)$ and $\gamma(\tau_1, \tau_3)$, and drop $-(\tau_1 + \tau_2 + \tau_3)$ terms in (18).
Figure 5: Welfare $\tilde{\mathcal{W}}(\mu; \tau)$ and the inflation rate

Notice that $u^{-1} \left( \frac{\mu}{\beta} \right)$ and $\frac{\beta k}{\mu}$ are less than $x^*$ for any $\mu \geq \beta$, so both $\tilde{\mathcal{W}}_n c(\mu; \tau)$ and $\tilde{\mathcal{W}}_t c(\mu; \tau)$ are decreasing in $\mu$. Next, by definition of $\mu_k$, $\tilde{\mathcal{W}}_n c(\mu_k; \tau) = \tilde{\mathcal{W}}_t c(\mu_k; \tau)$. Given these two observations, we get the left panel of Figure 5 that represents $\tilde{\mathcal{W}}(\mu; \tau)$ as a function of $\mu$ when $\gamma \geq \hat{\gamma}(\beta, k)$.

The story becomes richer when $\gamma < \hat{\gamma}(\beta, k)$. In this case, the economy can be in all types of equilibria depending on $\mu$. By the similar arguments above, $\tilde{\mathcal{W}}(\mu; \tau)$ is monotonically decreasing in $\mu$ for all $\mu \geq \mu_\gamma$. Substituting the expression of $\alpha$ and $\pi$ from proposition 1 into $\tilde{\mathcal{W}}(\mu; \tau)$, we obtain

$$\tilde{\mathcal{W}}_c(\mu; \tau) = u(x_\gamma) - x_\gamma - \frac{\mu}{\beta} \frac{\gamma x_\gamma}{x_\gamma - \gamma}$$

for all $\mu \in [\beta, \mu_\gamma)$.

First, taking derivatives the above expression with respect to $\mu$ and solving, we get

$$\frac{\partial \tilde{\mathcal{W}}_c}{\partial \mu} = \frac{\mu - \beta x_\gamma}{\beta} - \frac{\gamma x_\gamma}{(x_\gamma - \gamma)} < 0,$$

so $\tilde{\mathcal{W}}(\mu; \tau)$ is decreasing in $\mu$ for all $\mu \in [\beta, \mu_\gamma)$. Second, by using the expression $-\frac{\mu}{\beta} \frac{x_\gamma^2}{x_\gamma - \gamma} + u(x_\gamma) = -k + u \left( \frac{\beta k}{\mu_\gamma} \right)$, from (17), we obtain

$$\tilde{\mathcal{W}}_c(\mu_\gamma; \tau) - \tilde{\mathcal{W}}_t c(\mu_\gamma; \tau) = \frac{\mu_\gamma - \beta}{\beta} \left( x_\gamma - \frac{\beta k}{\mu_\gamma} \right) > 0$$

because $x_\gamma > \frac{\beta k}{\mu_\gamma}$. Thus, there is a jump in $\tilde{\mathcal{W}}(\mu; \tau)$ at $\mu = \mu_\gamma$ as depicted in the right panel of
It seems worthwhile to spend a little time on the multiplicity of equilibrium when \( \gamma = \hat{\gamma}(\beta, k) \). In this knife edge case, we simply assumed that the economy is in the threat of counterfeiting equilibrium in propositions 1 and 2. However, in principle both equilibria are possible, and more interestingly, given the same counterfeiting environment and the same inflation rate at \( \mu_\gamma \), the economy with counterfeiting achieves higher welfare than the one where counterfeits do not circulate. This is because the economy that admits counterfeiting supports greater trade size in the \( DM \). However, one should not interpret this result as that counterfeiting improves welfare by working as liquidity provision. Counterfeiting occurs as a strategic outcome. Welfare is higher if counterfeiting was not possibility.

Perhaps the most interesting result in Figure 5 is that \( \tilde{W}(\mu; \tau) \) is monotonically decreasing in \( \mu \) for all cases: \( \gamma \geq \hat{\gamma}(\beta, k) \). Thus, optimal monetary policy is the Friedman rule independent of anti-counterfeiting policy, which is re-emphasized as the next proposition.

**Proposition 3** Optimal monetary policy is the Friedman rule independent of anti-counterfeiting policy: \( \mu^*(\tau) = \beta \) for all \( \tau \in \mathbb{R}^3_+ \).

The proposition that the Friedman rule is optimal is very robust in monetary theory.\(^{25}\) Conventional logic for this statement is that the Friedman rule makes inter-temporal cost of holding money zero so it maximizes the trade surplus, \( u(x) - x \), in the \( DM \) by supporting the efficient amount of trade, \( x^* \). In our model, when \( \gamma \geq \hat{\gamma}(\beta, k) \) and hence counterfeiting does not occur for all \( \mu \geq \beta \), the Friedman rule is optimal in this conventional way.

The rationale behind the Friedman rule, however, is quite different when \( \gamma < \hat{\gamma}(\beta, k) \). In the counterfeiting equilibrium, a decrease in inflation increases the quantity of goods traded in the \( DM \), \( x_\gamma \), by reducing the cost of holding money. Furthermore, welfare also increases as inflation falls (see the right panel of Figure 5). Thus, is the Friedman rule optimal because it maximizes the trade surplus? The consumption in the \( DM \) when \( \mu = \beta \) is strictly higher than the efficient level

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\(^{25}\)There are some papers, however, that study conditions where deviation from the Friedman rule is optimal in monetary models (see, e.g., Sanches and Williamson 2010).
\( x^* \) (see lemma 4), which implies \( x_\gamma > x^* \) with \( \mu \approx \beta \). Thus, lowering \( \mu \) near \( \beta \) decreases the trade surplus, \( u(x_\gamma) - x_\gamma \). Then, why does the Friedman rule maximize welfare here? This is because as the inflation rate \( \mu \) falls, the measure of buyers who commit forgery decreases as described in Figure 4, so does the welfare costs from counterfeiting.

We are now ready to find the optimal government policy \((\mu^*, \tau^*)\) by maximizing \( W(\mu^*(\tau), \tau) \).

By proposition 3, it suffices to let \( \mu^*(\tau) = \beta \) to find \( \tau^* \). However, to obtain more general idea, we take the monetary policy \( \mu \) as given and find \( \tau^*(\mu) \) in a similar way to derive \( \mu^*(\tau) \). To analyze the effects of \( \tau \) on welfare, one has to consider the agents’ optimal behaviors that determine the equilibrium type. More precisely, the maximized welfare given \( \mu \) is \( W^*(\tau; \mu) = \max \{ W^{tc*}(\mu), W^{c*}(\mu) \} \) where

\[
(\text{Ptc}) \quad W^{tc*}(\mu) = \max_{\tau \in \mathbb{R}^3_+} \left\{ u \left( \frac{\beta k(\tau_1, \tau_2)}{\mu} \right) - \frac{\beta k(\tau_1, \tau_2)}{\mu} - \tau_1 - \tau_2 - \tau_3 \right\}
\]

subject to

\[
(19) \quad u \left( \frac{\beta k(\tau_1, \tau_2)}{\mu} \right) - k(\tau_1, \tau_2) \geq u(x_\gamma) - \frac{\mu}{\beta} \frac{x_\gamma^2}{x_\gamma - \gamma}
\]

and

\[
(\text{Pc}) \quad W^{c*}(\mu) = \max_{\tau \in \mathbb{R}^3_+} \left\{ u(x_\gamma) - x_\gamma - \frac{\mu}{\beta} \frac{\gamma x_\gamma}{x_\gamma - \gamma} - \tau_1 - \tau_2 - \tau_3 \right\}
\]

subject to

\[
(20) \quad u(x_\gamma) - \frac{\mu}{\beta} \frac{x_\gamma^2}{x_\gamma - \gamma} \geq u \left( \frac{\beta k(\tau_1, \tau_2)}{\mu} \right) - k(\tau_1, \tau_2).
\]

The constraint (19) is the condition for the economy to be in the threat of counterfeiting equilibrium whereas the constraint (20) is the condition for the existence of the counterfeiting equilibrium.

Let \( \tau^{tc}(\mu) \) and \( \tau^c(\mu) \) be a solution of \((\text{Ptc})\) and \(\text{(Pc)}\) respectively. Then, optimal anti-counterfeiting
policy given \( \mu, \tau^*(\mu) \), is

\[
\tau^*(\mu) = \begin{cases} 
\tau^{tc}(\mu) & \text{if } \mathcal{W}^*_t(\mu) \geq \mathcal{W}^*_c(\mu) \\
\tau^c(\mu) & \text{if } \mathcal{W}^*_t(\mu) < \mathcal{W}^*_c(\mu) 
\end{cases}
\]

Whether \( \mathcal{W}^*_t(\mu) \gtrless \mathcal{W}^*_c(\mu) \) and the form of \( \tau^*(\mu) \) depend on particular functions of \( k(\tau_1, \tau_2), \gamma(\tau_1, \tau_3), \) and \( u(x) \). However, a certain rule of optimal anti-counterfeiting policy can be found by making two observations. First, welfare under the threat of counterfeiting equilibrium depends on the counterfeiting cost but not on the screening cost. Lowering the screening cost only makes the constraint (19) tighter. Thus, any solution to \( P_{tc} \) must feature \( \tau_3 = 0 \) because the objective function can be increased otherwise. Second, welfare under the counterfeiting equilibrium, on the other hand, depends on the screening cost but not on the counterfeiting cost. Increasing the counterfeiting cost only tightens the constraint (20) which implies that it must be \( \tau_2 = 0 \) to solve \( P_c \). Therefore, the government should focus on either increasing the counterfeiting cost or decreasing the screening cost, so the optimal anti-counterfeiting policy is dichotomous which is formalized in the next proposition.

**Proposition 4** If \( \tau^*(\mu) = \tau^{tc}(\mu) \), then \( \tau^*_3(\mu) = 0 \), whereas if \( \tau^*(\mu) = \tau^c(\mu) \), then \( \tau^*_2(\mu) = 0 \).

After deriving \( \tau^*(\mu) \), the optimal government policy is simply given by \( (\beta, \tau^*(\beta)) \) by virtue of proposition 3. The structure of \( (\beta, \tau^*) \) hinges on a form of cost functions and agent’s preference. For example, if \( \lim_{\tau_1 \to 0} k_1 = \infty \) and \( \lim_{\tau_1 \to 0} \gamma_1 = -\infty \), then the optimal government policy \( (\beta, \tau^*) \) features \( \tau^*_1 > 0 \) in all cases, \( \mathcal{W}^*_t \gtrless \mathcal{W}^*_c \), which means that security features must always be incorporated into banknotes. Observe that whenever \( \tau^* = \tau^c(\beta) \), counterfeits circulate in the economy under the optimal government policy.
5 Conclusion

In this paper, we have constructed a search theoretic model of money to study how counterfeiting affects economic activity and optimal government policy. We have shown that there are three types of equilibria when an economy is susceptible to counterfeiting. When the cost of counterfeiting is high enough, the incentive to produce fake money does not exist. When the counterfeiting cost is not too high but the screening cost is relatively high, then the potential threat of counterfeiting generates a resalability constraint on money thereby affecting the allocation, but there is still no counterfeits in equilibrium. Finally, when both the counterfeiting cost and the screening cost are sufficiently low, counterfeiting occurs in equilibrium. We also show that it is more likely, in this economy, that counterfeiting occurs when inflation is low because of its effects on the value of money.

We also used the model to provide implications for government policy. First, the Friedman rule is optimal independent of anti-counterfeiting policy. When counterfeits do not exist in equilibrium, the Friedman rule is optimal because it maximizes the trade surplus. We add new insight here: When counterfeits do exist in equilibrium, the Friedman rule is optimal not because it maximizes the trade surplus but because it minimizes the welfare cost of counterfeiting. The trade surplus is not maximized with the Friedman rule in this case. Second, the model suggests that anti-counterfeiting policy must focus on improvement along one of two dimensions—increasing the counterfeiting cost or decreasing the screening cost—but not both.

A natural extension of our analysis would be to relax some of the assumptions of our model. For example, money is the only possible medium of exchanges in the model. In practice, however, other financial assets, such as government bonds and asset backed securities that are also subject to moral hazard problems, are widely used as a medium of exchanges. It would be interesting to introduce our description of screening into the counterfeiting model with multiple real assets, such as in Li, Rocheteau, and Weill (2012), and study how anti-counterfeiting policy for a particular asset or the supply of safe assets that are immune to faking into the economy affects counterfeiting of other types of assets. Another potentially interesting extension would be to assume that the
central bank determines the rate of confiscation of counterfeits in contrast to 100% confiscation in our model, and to study how this decision affects counterfeiting and welfare.

References


Appendix: Proofs

Proof of Lemma 1. Suppose there exists an equilibrium in which the buyer accumulates $m_t$ units of legal money and $m_c^t$ units of counterfeits both in terms of $CM$ goods of period $t+1$, makes the offer $(x_t, d_t)$, and hands over $\hat{d}_t > 0$ units of genuine money and $\hat{d}_c^t > 0$ units of legal money where $\hat{d}_t + \hat{d}_c^t = d_t$. Then, given $u(x_t^s) - \beta \hat{d}_t^s = 0$, the buyer’s payoff is

$$S_t^b = - \left( \frac{\phi_t}{\phi_{t+1}} - \beta \right) m_t - k + I_{\{r=A\}} (1 - I_{\{s=Y\}}) \left[ u(x_t) - \beta \hat{d}_t \right].$$

Now consider another set of actions such that $m' = m_t - \hat{d}_t$, $\hat{d}' = 0$, and $\hat{d}_c^t = d_t$, and the buyer makes the same offer $(x_t, d_t)$. The buyer’s payoff with this choice of actions is

$$S_t^{b'} = S_t^b + \left\{ \frac{\phi_t}{\phi_{t+1}} - \beta \left[ 1 - I_{\{r=A\}} (1 - I_{\{s=Y\}}) \right] \right\} \hat{d}_t > S_t^b,$$

so we get contradiction. Thus, it must be $d_t = \hat{d}_t$ or $d_t = \hat{d}_c^t$. The proof of remaining parts is straightforward.

Proof of Lemma 2. We first prove that for any solution to $(P)$, $\pi < 1$. Suppose $\pi = 1$ which requires $\alpha \in (0, 1)$ by (5). Then, substituting the indifference condition of (4), $[\mu - (1 - \eta)\beta] d =
and (5), and attains higher value of the objective function. Thus, the only possible case with $\pi = 0$. Then, the indifference condition to have $\alpha \in (0, 1)$ if and only if $\pi \in (0, 1)$. The “if” part is explained above so we focus on proving the “only if” part by showing a contradiction otherwise. Suppose there exists a solution $\pi = \alpha x d$ by showing a contradiction otherwise. Suppose there exists a solution $\pi = \alpha x_{\alpha}$ that satisfies all constraints in (4) and (5), and attains higher value of the objective function than $(x, d, \alpha, \eta, \pi)$, a contradiction.

Proof of Lemma 3. 1) We first prove that $\eta = 1$. To find a contradiction, suppose there exists a solution $(x, d, \eta, \alpha, \pi)$ of $(P)$ with $\eta < 1$. Since $\eta = \eta x d < \gamma$, the indifference condition to have $\alpha \in (0, 1)$ is $\alpha < 1$ but $\pi = 0$. Then, the indifference condition to have $\alpha \in (0, 1)$ and $\eta = \eta x d$ is $\alpha \in \alpha d < d$, so $[\mu - (1 - \eta)\beta] d' < k$. Then, $(x, d', \alpha' = 1, \eta, \pi' = 0)$ satisfies constraints (4) and (5). However, the change in the objective function is

$$
\triangle S^b = \eta' u(x') - \eta u(x)
$$

$$
\approx (\eta + \varepsilon) \left\{ u(x) - u'(x) \frac{\beta x \varepsilon}{\mu - \beta + \beta (\eta + \varepsilon)} \right\} - \eta u(x) > 0
$$

with sufficiently small $\varepsilon > 0$. Therefore, for the solution with $\eta < 1$ to exist, it must be that $\alpha \in (0, 1)$ and $\pi \in (0, 1)$. Now consider $\eta' = \eta + \varepsilon$ and $\pi' = \pi - \varepsilon$ where $\varepsilon = \frac{\beta d e}{\mu - \beta + (\eta + \varepsilon)\beta}$, so $\lim_{\varepsilon \rightarrow 0} \varepsilon$. Let $\alpha' = 1$, $\eta' = 1$, $\pi' = 0$ satisfies constraints (4) and (5). However, the change in the objective function is

$$
\triangle S^b = \eta' u(x') - \eta u(x)
$$

$$(\eta + \varepsilon) \left\{ u(x) - u'(x) \frac{\beta x \varepsilon}{\mu - \beta + \beta (\eta + \varepsilon)} \right\} - \eta u(x) > 0
$$

with sufficiently small $\varepsilon > 0$. Therefore, for the solution with $\eta < 1$ to exist, it must be that $\alpha \in (0, 1)$ and $\pi \in (0, 1)$. Now consider $\eta' = \eta + \varepsilon$ and $\pi' = \pi - \varepsilon$ where $\varepsilon = \frac{\beta d e}{\mu - \beta + (\eta + \varepsilon)\beta}$. Since $\eta \in (0, 1)$ and $\pi \in (0, 1)$, there exists sufficiently small $\varepsilon > 0$ such that $\eta' \in (0, 1)$ and $\pi' \in (0, 1)$. Then, $(x, d, \eta', \alpha, \pi')$ satisfies all constraints in $(P)$ but delivers higher value of the objective function, a contradiction.

2) If $x > \alpha d$, then $\eta = 0$ given the result of lemma 2. Thus, to prove $x = \alpha d$, suppose there exists a solution $(x, d, \eta, \alpha, \pi)$ of $(P)$ with $x < \alpha d$. Assume $\alpha = 1$. Then, there exists $d' < d$ such that $x < \beta d'$. Then, $(x, d', \eta, \alpha, \pi)$ satisfies (4) and (5) but attains higher value of the objective function. Thus, the only possible case with $x < \alpha d$ is that $\eta = (0, 1)$ and $\pi \in (0, 1)$. In this
Proof of Proposition 1. Since we already characterized a solution of \((P)\), we only prove that equilibrium can be characterized by solving \((P)\). Given a solution \((x, d, \alpha, \eta, \pi)\), suppose \(\{\alpha, [\eta, \pi]\}\) is a unique Pareto dominant Nash equilibrium of the subgame \(\Gamma(x,d)\). Then \((x, d, \alpha, \eta, \pi)\) is an equilibrium outcome by construction of \((P)\). Also, any equilibrium must solve \((P)\) because the buyer can attain higher payoff by posting \((x, d)\) that solves \((P)\) otherwise. Thus, it suffices to show that if \((x, d, \alpha, \eta, \pi)\) is a solution to \((P)\), then \(\{\alpha, [\eta, \pi]\}\) is the unique Pareto dominant Nash equilibrium of \(\Gamma(x,d)\).

Suppose \((x, d, \alpha, \eta, \pi)\) solves \((P)\). As argued above, \(x > 0\) and \(\eta = 1\). Thus, any Nash equilibrium \(\{\alpha', [\eta', \pi']\}\) of \(\Gamma(x,d)\) with \(\alpha' = 0\) is Pareto dominated by \(\{\alpha, [\eta, \pi]\}\) because it must be that \(\eta' = 0\). Since any solution to \((P)\) has either \(\alpha = 1\) or \(\alpha \in (0, 1)\), we consider each case separately.

1) First, suppose that \(\alpha = 1\) so \(\pi = 0\). Thus, it must be \(\mu d \leq k\) and \(x = \beta d\). Consider any Nash equilibrium \(\{\alpha', [\eta', \pi']\} \neq \{\alpha, [\eta, \pi]\}\) of \(\Gamma(x,d)\). i) Assume \(\alpha' = 1\). Then, the only Nash equilibrium, if it exists, is with \(\eta' < 1\), so \(\{\alpha', [\eta', \pi']\}\) is Pareto dominated by \(\{\alpha, [\eta, \pi]\}\). ii)
Assume $\alpha' \in (0, 1)$ which requires $\mu d - (1 - \eta')\beta d = k + \eta' \pi' u(x)$ by (4). This is possible only if $\mu d = k$ with $\eta' = 1$ and $\pi' = 0$. However, if $\pi' = 0$, then it must be $\eta' = 0$ because $-x + \alpha' \beta d < 0$, a contradiction. Therefore, if $\alpha = 1$ and $\pi = 0$, then $\{\alpha, [\eta, \pi]\}$ is the unique Pareto dominant Nash equilibrium of the subgame $\Gamma(x, d)$.

2) Second, suppose that $\alpha \in (0, 1)$ and $\pi \in (0, 1)$. Then, it must be $\mu d = k + \pi u(x), (1 - \alpha)x = \gamma$, and $x = \alpha \beta d$. Now consider any Nash equilibrium $\{\alpha', [\eta', \pi']\} \neq \{\alpha, [\eta, \pi]\}$ of $\Gamma(x, d)$. i) Suppose $\alpha' \in (\alpha, 1]$. Then, the best response of the seller is $\eta' = 1$ and $\pi' = 0$. But the buyer’s best response is, then, $\alpha' = 0$, because $\mu d > k$, a contradiction. ii) Consider the case that $\alpha' = \alpha$. If $\eta' < 1$, $\{\alpha', [\eta', \pi']\}$ is Pareto dominated by $\{\alpha, [\eta, \pi]\}$, and hence assume $\eta' = 1$. However, the only Nash equilibrium with $\alpha' = \alpha$ and $\eta' = 1$ is with $\pi' = \pi$, a contradiction. iii) Finally, assume $\alpha' \in (0, \alpha)$. By the same reason above, assume $\eta' = 1$. Then, it must be $\alpha' \beta d - \alpha' x - \gamma \geq 0$ with $\pi' = 1$. Otherwise $\eta' = 0$ because $\alpha' < \alpha$. However, if $(\eta', \pi') = (1, 1)$, then $\alpha' = 1$ because $\mu d < k + u(x)$, a contradiction. Therefore, if $\alpha \in (0, 1)$ and $\pi \in (0, 1)$, then $\{\alpha, [\eta, \pi]\}$ is the unique Pareto dominant Nash equilibrium of the subgame $\Gamma(x, d)$. ■